# Exact Price of Anarchy for Polynomial Congestion Games*


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**Abstract.** We show exact values for the price of anarchy of weighted and unweighted congestion games with polynomial latency functions. The given values also hold for weighted and unweighted *network* congestion games.



**Keywords.** congestion games, selfish routing, price of anarchy

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# 1 Introduction

## 1.1 Motivation and Framework

Large-scale communication networks like, e.g., the Internet often lack a central regulation for several reasons: The size of the network may be too large, or the users may be free to act according to their private interests. Even cooperation among the users may be impossible due to the fact that users may not even know each other. Such an environment – where users neither obey some central control instance nor cooperate with each other – can be modeled as a *non-cooperative game* [22].

One of the most widely used solution concepts for non-cooperative games is the concept of *Nash equilibrium*. A Nash equilibrium is a state in which no player can improve his objective by unilaterally changing his strategy. We call a Nash equilibrium *pure* if every player chooses a single (pure) strategy, *mixed* if all players make a random choice according to (stochastically independent) probability distributions over their pure strategies, and *correlated* if all players randomize according to some joint (correlated) probability distribution [2].

Rosenthal [28] introduced a special class of non-cooperative games, now widely known as *congestion games*. These games provide a simple, yet powerful model of competitive *resource sharing*: There are a finite number of players and finitely many resources, and the strategies available to the players consist of subsets of the given resources. Since the players' strategies may overlap, congestion effects occur and strategic interaction arises. Each resource experiences a *latency* that depends on the total number of players using this resource – and each player is incurred a *private cost* that is defined as the sum of the latencies over his chosen resources.

Milchtaich [23] generalized congestion games to *weighted congestion games* in which the players have weights and thus different influence on the congestion of the resources. Weighted congestion games provide us with a general framework for modeling any kind of non-cooperative resource sharing problem. A typical resource sharing problem is routing. In a routing game, the strategy sets of the players correspond to paths in a network. Routing games where the demand of the players cannot be split among multiple paths are also called *(weighted) network congestion games*. Another model for selfish routing – the so called *Wardrop* model – was already studied in the 1950's (see, e.g., [5, 32]) in the context of road traffic systems, where traffic flows can be split arbitrarily. The Wardrop model can be seen as a special network congestion game with infinitely many players who each have merely infinitesimal weight. For this reason, these games are sometimes called *non-atomic congestion games* [31].

In order to measure the degradation of social welfare due to the selfish behavior of the players, Koutsoupias and Papadimitriou [19] introduced a global objective function, usually termed *social cost*. They defined the *price of anarchy*, also called coordination ratio, as the worst-case ratio between the social cost in a Nash equilibrium and that in some social optimum. Thus, the price of anarchy measures the extent to which non-cooperation approximates cooperation. One typically differentiates between the pure, mixed, and correlated prices of anarchy, corresponding to Nash equilibria in pure, mixed, and correlated strategies.

The price of anarchy directly depends on the definition of social cost: In their seminal paper, Koutsoupias and Papadimitriou [19] considered a very simple weighted network congestion game on parallel links, now known as the KP-model. For this model, they defined the social cost as the expected maximum latency. For the Wardrop model, in contrast, Roughgarden and Tardos [30] considered social cost defined as the *total latency*, which is a measure for the (weighted) total travel time. For (weighted) congestion games (i.e., congestion games with a finite number of players with non-negligible demands), Awerbuch et al. [3] and Christodoulou and Koutsoupias [6] also considered the total latency. In this setting, they showed asymptotic bounds on the price of anarchy for weighted (and unweighted) congestion games with polynomial latency (cost) functions. Here, all polynomials are of maximum degree $d$ and have non-negative coefficients. For the case of linear latency functions they gave exact bounds on the price of anarchy.

### 1.2 Contribution and Comparison

In this work we prove *exact* bounds on the prices of anarchy for unweighted and weighted congestion games with polynomial latency functions. We use the total latency as social cost measure. This improves on results by Awerbuch et al. [3] and Christodoulou and Koutsoupias [6], where non-matching upper and lower bounds are given.

We now describe our findings in more detail.

- For *weighted congestion games* we show that the price of anarchy (PoA) is exactly

$$\Phi_d^{d+1},$$

where $\Phi_d$ is a natural generalization of the golden ratio to larger dimensions such that $\Phi_d$ is the unique non-negative real solution to $(x+1)^d = x^{d+1}$. This result closes the gap between the so far best upper and lower bounds of $O(2^d d^{d+1})$ and $\Omega(d^{d/2})$ from [3].

- For *unweighted congestion games* we show that the price of anarchy (PoA) is exactly

$$\frac{(k+1)^{2d+1} - k^{d+1}(k+2)^d}{(k+1)^{d+1} - (k+2)^d + (k+1)^d - k^{d+1}},$$

where $k = \lfloor \Phi_d \rfloor$. The PoA is bounded from below by $\lfloor \Phi_d \rfloor^{d+1}$ and from above, of course, by $\Phi_d^{d+1}$. Prior to this paper, the best known upper and lower bounds were shown to be of the form $d^{d(1-o(1))}$ [6]. However, the term $o(1)$ still hides a gap between the upper and the lower bound.

We show that the pure, mixed, and correlated prices of anarchy always coincide and that the above values also hold for the subclasses of unweighted and weighted *network* congestion games.

For our upper bounds, we use a similar technique as Christodoulou and Koutsoupias [6], yet with a more elaborate analysis. The core of our analysis is to minimize $\frac{c_2}{1-c_1}$ under the constraint that

$$y \cdot f(x+1) \leq c_1 \cdot x \cdot f(x) + c_2 \cdot y \cdot f(y) \tag{1.1}$$

holds for all polynomials $f$ with positive coefficients and maximum degree $d$, and for all reals $x, y \geq 0$. For the case of unweighted players, it is sufficient if (1.1) holds for all integers $x, y$. In order to prove their upper bound, Christodoulou and Koutsoupias [6] looked at (1.1) with $c_1 = \frac{1}{2}$ and gave an asymptotic estimate for $c_2$. In our analysis, we optimize both parameters $c_1, c_2$. This optimization process requires new ideas and is non-trivial.

Table 1 shows a numerical comparison of our exact values for the price of anarchy with the previous results of Awerbuch et al. [3] and Christodoulou and Koutsoupias [6]. The lower bounds in the table stem from construction schemes given in the cited works, before any estimates are applied. Values in parentheses denote cases in which the lower bound for linear functions is larger than the coordination ratio in the respective instance that corresponds to $d$. The construction scheme in [3, Theorem 4.3] yields a price of anarchy approximating $\frac{1}{e}\sum_{k=1}^{\infty}\frac{k^d}{k!}$, which is the value of the $d$-th Bell number. In [6, Theorem 10], a network with price of anarchy $\frac{(N-1)^{d+2}}{N}$ is given, where $N$ is the largest integer for which $(N-1)^{d+2} \le N^d$ holds. The column with the upper bound from [6] is computed by using (1.1) with $c_1 = \frac{1}{2}$ and optimizing $c_2$ with help of our analysis. Thus, the column shows the best possible bounds that can be shown with $c_1 = \frac{1}{2}$.

**Table 1:** The price of anarchy – comparison of our results to Christodoulou and Koutsoupias [6], Awerbuch et al. [3], and Roughgarden and Tardos [31]

| | | Congestion Games | | | | | Wardrop Model |
|---|---|---|---|---|---|---|---|
| | | **Unweighted Players** | | | **Weighted Players** | | |
| $d$ | $\Phi_d$ | Exact | Upper Bound [6] | Lower Bound [6] | Exact | Lower Bound [3] | Exact [31] |
| 1 | 1.618 | **2.5** | 2.5 | 2.5 | **2.618** | 2.618 | 1.333 |
| 2 | 2.148 | **9.583** | 10 | (2.5) | **9.909** | (2.618) | 1.626 |
| 3 | 2.630 | **41.54** | 47 | (2.5) | **47.82** | 5 | 1.896 |
| 4 | 3.080 | **267.6** | 269 | 21.33 | **277.0** | 15 | 2.151 |
| 5 | 3.506 | **1,514** | 2,154 | 42.67 | **1,858** | 52 | 2.394 |
| 6 | 3.915 | **12,345** | 15,187 | 85.33 | **14,099** | 203 | 2.630 |
| 7 | 4.309 | **98,734** | 169,247 | 170.7 | **118,926** | 877 | 2.858 |
| 8 | 4.692 | **802,603** | 1,451,906 | 14,762 | **1,101,126** | 4,140 | 3.081 |

### 1.3 Related Work

The papers most closely related to our work are those of Awerbuch et al. [3] and Christodoulou and Koutsoupias [6, 7]. For (unweighted) congestion games and social cost defined as average private cost (which in this case is the same as total latency) it was shown that the price of anarchy of pure Nash equilibria is $\frac{5}{2}$ for linear latency functions and $d^{\Theta(d)}$ for polynomial latency functions of maximum degree $d$ [3, 6]. The bound of $\frac{5}{2}$ for linear latency functions also holds for the correlated and thus also for the mixed price of anarchy [7]. For *weighted* congestion games the mixed price of anarchy for total latency is $\frac{3+\sqrt{5}}{2}$ for linear latency functions and $d^{\Theta(d)}$ for polynomial latency functions [3].

The *price of anarchy* [27], also known as *coordination ratio*, was first introduced and studied by Koutsoupias and Papadimitriou [19]. As a starting point of their investigation they considered a simple weighted congestion game on parallel links, now known as KP-model. In the KP-model, latency functions are linear and social cost is defined as the maximum expected congestion on a link. In this setting, there exist *tight* bounds on the price of anarchy of $\Theta(\frac{\log m}{\log\log m})$ for identical links [8, 20] and $\Theta(\frac{\log m}{\log\log\log m})$ for related links [8]. The price of anarchy has also been studied for variations of the KP-model, namely for non-linear latency functions [9, 17], for the case of restricted strategy sets [4, 14], for the case of incomplete information [18], and for different social cost measures [16, 21, 13]. In particular, Lücking et al. [21] study the total latency (they call it quadratic social cost) for routing games on parallel links with linear latency functions. For this model they show that the price of anarchy is exactly $\frac{4}{3}$ for the case of identical player weights and $\frac{9}{8}$ for the case of identical links and arbitrary player weights.

The class of *congestion games* was introduced by Rosenthal [28] and extensively studied afterwards (see, e.g., [12, 23, 24]). In Rosenthal's model, the strategy of each player is a subset of resources. Resource latency functions can be arbitrary but they only depend on the number of players sharing the same resource. Rosenthal showed that such games always admit a pure Nash equilibrium using a potential function. Monderer and Shapley [24] characterized games that possess a potential function as potential games and show their relation to congestion games. Milchtaich [23] considered weighted congestion games with player specific payoff functions and showed that these games do not admit a pure Nash equilibrium in general. Fotakis et al. [12, 11] considered the price of anarchy for symmetric weighted network congestion games in layered networks [12] and for symmetric (unweighted) network congestion games in general networks [11]. In both cases they defined social cost as expected maximum latency. For a survey on weighted congestion games we refer to [15].

Inspired by the arisen interest in the price of anarchy, Roughgarden and Tardos [30, 31] re-investigated the Wardrop model and used the *total latency* as a social cost measure. In this context, the price of anarchy was shown to be $\frac{4}{3}$ for linear latency functions [30] and $(1-d\cdot(d+1)^{-(d+1)/d})^{-1}$ for polynomial latency functions of maximum degree $d$, i.e., $\Theta(\frac{d}{\log d})$ as $d\to\infty$ [31]. An overview on results for this model can be found in [29].

### 1.4 Roadmap

The rest of this paper is organized as follows. In Section 2, we give an exact definition of (weighted) congestion games. We establish exact bounds on the price of anarchy for weighted congestion games in Section 3 and for unweighted congestion games in Section 4. All purely analytical tools that we believe to be of independent interest are given in Section 5.

## 2 Notation

**General** For all $d\in\mathbb{N}$, let $[d] := \{1,\dots,d\}$ and $[d]_0 := [d]\cup\{0\}$. For a vector $\boldsymbol{v} = (v_1,\dots,v_n)$, let $(\boldsymbol{v}_{-i}, v_i') := (v_1,\dots,v_{i-1},v_i',v_{i+1},\dots,v_n)$. Moreover, we denote by $B_d$ the $d$-th Bell Number and by $\Phi_d$ a natural generalization of the golden ratio such that $\Phi_d$ is the (only) positive real solution to $(x+1)^d = x^{d+1}$. We remark that $\Phi_d$ is irrational for all $d\in\mathbb{N}$ (see Theorem 9). The asymptotic growth of $\Phi_d$ is $\Theta(\frac{d}{\ln d})$ (see Theorem 10).

**Congestion Games** A *(weighted) congestion game* $\Gamma$ is a tuple

$$\Gamma = \left(n, E, (w_i)_{i\in[n]}, (\mathscr{S}_i)_{i\in[n]}, (f_e)_{e\in E}\right) .$$

Here, $n$ is the number of *players* (or *users*) and $E$ is a finite set of *resources*. Each player $i \in [n]$ is endowed with a *weight* $w_i \in \mathbb{R}_{>0}$ and a *strategy set* $\mathscr{S}_i \subseteq 2^E$, which contains sets of resources. We call a congestion game *unweighted* if the weights of all players are 1. Moreover, we define $\mathscr{S} := \mathscr{S}_1 \times \ldots \times \mathscr{S}_n$ as the space of pure *strategy profiles*. Associated with every resource $e \in E$ is a *latency function* $f_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that maps the total weight of all players using resource $e$ to the experienced *latency* on $e$. For $d \in \mathbb{N}$, we denote by $\mathcal{P}_d$ the set of all *polynomial latency functions* with maximum degree $d$ and non-negative coefficients. That is, for all $f \in \mathcal{P}_d$ there are $a_0, \ldots, a_d \in \mathbb{R}_{\geq 0}$ such that $f(x) = \sum_{j=0}^{d} a_j \cdot x^j$. In this work, we only consider games with latency functions from $\mathcal{P}_d$.

For any arbitrary set of non-negative functions $\mathcal{F}$, we denote by $\mathcal{W}(\mathcal{F})$ the set of all weighted and by $\mathcal{U}(\mathcal{F})$ the set of all unweighted congestion games, respectively, where all latency functions belong to $\mathcal{F}$.

**Strategies and Strategy Profiles** A *pure strategy* for player $i \in [n]$ is some specific $s_i \in \mathscr{S}_i$, whereas a *randomized strategy* $S_i \in \Delta(\mathscr{S}_i)$ is a probability distribution over $\mathscr{S}_i$ where $S_i(s_i)$ denotes the probability that player $i$ chooses the pure strategy $s_i$.

A *pure strategy profile* is an $n$-tuple $\boldsymbol{s} = (s_1, \ldots, s_n) \in \mathscr{S}$ whereas a *correlated strategy profile* $\boldsymbol{P} \in \Delta(\mathscr{S})$ is a probability distribution over $\mathscr{S}$. We denote by $\boldsymbol{P}(\boldsymbol{s})$ the probability that the players play the pure profile $\boldsymbol{s}$. Clearly, $\boldsymbol{P}$ *induces* for each player $i \in [n]$ a randomized strategy $P_i \in \Delta(\mathscr{S}_i)$ by $P_i(s_i') = \sum_{\boldsymbol{s} \in \mathscr{S} | s_i = s_i'} \boldsymbol{P}(\boldsymbol{s})$. When the probability distributions of the players are stochastically independent, i.e.,

$$\boldsymbol{P}(\boldsymbol{s}) = \prod_{i \in [n]} P_i(s_i)\,,$$

we say that $\boldsymbol{P}$ is a *mixed strategy profile*. For $s_i' \in \mathscr{S}_i$, we denote by $(\boldsymbol{P}_{-i}, s_i') =: \boldsymbol{P}'$ the correlated profile where the players play each pure strategy profile $\boldsymbol{t} \in \mathscr{S}$ with probability $\boldsymbol{P}'(\boldsymbol{t}) = (\boldsymbol{P}_{-i}, s_i')(\boldsymbol{t}) := \sum_{s_i \in \mathscr{S}_i} \boldsymbol{P}(\boldsymbol{t}_{-i}, s_i)$ if $t_i = s_i'$ and $\boldsymbol{P}'(\boldsymbol{t}) := 0$ otherwise. For ease of notation, we identify each pure strategy $s_i \in \mathscr{S}_i$ with the (degenerate) randomized strategy and each pure strategy profile $\boldsymbol{s} \in \mathscr{S}$ with the (degenerate) correlated profile that assign all probability to $s_i$ and $\boldsymbol{s}$, respectively.

**Private Cost** Let $\boldsymbol{s} \in \mathscr{S}$ be a pure strategy profile. For any resource $e \in E$, we define a *load function* $l_e : \mathscr{S} \to \mathbb{R}_{\geq 0}$ by $l_e(\boldsymbol{s}) := \sum_{i \in [n] | e \in s_i} w_i$. Then, the *private cost* of player $i \in [n]$ is defined by $\mathrm{PC}_i : \Delta(\mathscr{S}) \to \mathbb{R}_{\geq 0}$,

$$\mathrm{PC}_i(\boldsymbol{S}) := \sum_{\boldsymbol{s} \in \mathscr{S}} \boldsymbol{S}(\boldsymbol{s}) \sum_{e \in s_i} f_e(l_e(\boldsymbol{s}))\,.$$

**Social Cost** We use an objective function called *social cost* as a measure of social welfare. It is defined as the expected total latency. That is, $\mathrm{SC} : \Delta(\mathscr{S}) \to \mathbb{R}_{\geq 0}$,

$$\begin{aligned}\mathrm{SC}(\boldsymbol{S}) &:= \sum_{\boldsymbol{s}\in\mathscr{S}} \boldsymbol{S}(\boldsymbol{s}) \sum_{e\in E} l_e(\boldsymbol{s}) \cdot f_e(l_e(\boldsymbol{s})) \\ &= \sum_{\boldsymbol{s}\in\mathscr{S}} \boldsymbol{S}(\boldsymbol{s}) \sum_{i\in[n]} \sum_{e\in s_i} w_i \cdot f_e(l_e(\boldsymbol{s})) \\ &= \sum_{i\in[n]} w_i \cdot \mathrm{PC}_i(\boldsymbol{S})\,.\end{aligned}$$

The *optimum* associated with a weighted congestion game is defined by $\mathrm{OPT} := \min_{\boldsymbol{s}\in\mathscr{S}} \mathrm{SC}(\boldsymbol{s})$.

**Nash Equilibria and Price of Anarchy** We are interested in a special class of strategy profiles called Nash equilibria [25, 26] and correlated equilibria [2]. We give brief definitions here. A pure/mixed strategy profile $\boldsymbol{P}$ is a *pure/mixed Nash equilibrium* if and only if no player can improve his private cost by unilaterally changing his strategy. That is, for all players $i \in [n]$ and all pure strategies $s_i \in S_i$ the *Nash conditions* $\mathrm{PC}_i(\boldsymbol{P}) \leq \mathrm{PC}_i(\boldsymbol{P}_{-i}, s_i)$ hold. Note here that if the Nash conditions hold for all pure strategies $s_i \in S_i$, then they also holds for all mixed strategies over $S_i$.

A correlated profile $\boldsymbol{P}$ is a *correlated equilibrium* if and only if no player can improve his private cost by unilaterally replacing his pure strategies (but otherwise leaving the probability distribution over the strategy profiles unchanged). Formally, it holds for all players $i \in [n]$ and all pure strategy replacements $\delta_i : \mathscr{S}_i \to \mathscr{S}_i$ that

$$\mathrm{PC}_i(\boldsymbol{P}) \leq \sum_{\boldsymbol{s}\in\mathscr{S}} \boldsymbol{P}(\boldsymbol{s})\, \mathrm{PC}_i(\boldsymbol{s}_{-i}, \delta_i(s_i))\,.$$

It is easy to see that a correlated equilibrium satisfies in particular the *Nash conditions*. Conversely, each Nash equilibrium is also a correlated equilibrium.

The (correlated) *price of anarchy*, also called *coordination ratio* and denoted PoA, is defined for classes of congestion games. It is the worst-case (relative) loss in social welfare due to the selfish behavior of the players. Formally, when denoting the set of correlated equilibria by $\mathscr{E}^{\Gamma}_{\mathrm{corr}} \subseteq \Delta(\mathscr{S})$, define

$$\mathrm{PoA}_{\mathrm{corr}}(\mathcal{G}) := \sup_{\Gamma\in\mathcal{G}} \sup_{\boldsymbol{P}\in\mathscr{E}^{\Gamma}_{\mathrm{corr}}} \frac{\mathrm{SC}^{\Gamma}(\boldsymbol{P})}{\mathrm{OPT}^{\Gamma}}\,.$$

Here, we denote with a superscript $\Gamma$ the definitions for the respective game $\Gamma$. For ease of notation, we otherwise omit this superscript throughout the paper when it is unambiguous to do so. The pure and mixed prices of anarchy are defined accordingly. Clearly, $\mathrm{PoA}_{\mathrm{pure}}(\mathcal{G}) \leq \mathrm{PoA}_{\mathrm{mixed}}(\mathcal{G}) \leq \mathrm{PoA}_{\mathrm{corr}}(\mathcal{G})$.

## 3 Weighted Congestion Games

In this section, we establish the exact price of anarchy for weighted congestion games with polynomial latency functions of degree at most $d$.

### 3.1 Upper Bound

The upper bound for the price of anarchy crucially relies on the following Theorem 1. Since its statement is purely analytical and possibly of independent interest, we defer its proof to Section 5.3.

**Theorem 1.** *Let $d \in \mathbb{N}$. Then*

$$\min_{(c_1,c_2)\in(0,1)\times\mathbb{R}} \left\{ \frac{c_2}{1-c_1} \,\middle|\, \forall x,y \in \mathbb{R}_{\geq 0}, f \in \mathcal{P}_d : y \cdot f(x+y) \leq c_1 \cdot x \cdot f(x) + c_2 \cdot y \cdot f(y) \right\} = \Phi_d^{d+1}\,.$$

We remark that in Corollary 2 of Section 5.3, we also give concrete values for $c_1$ and $c_2$ so that the minimum is attained. In particular, this shows that the minimum on the left-hand side of Theorem 1 exists and is well-defined.

**Theorem 2.** *Let $d \in \mathbb{N}$. Then,* $\mathrm{PoA}_{\mathrm{corr}}(\mathcal{W}(\mathcal{P}_d)) \leq \Phi_d^{d+1}$.

*Proof.* Let $\Gamma$ be an arbitrary unweighted congestion game, $\boldsymbol{P} \in \Delta(\mathscr{S})$ be a correlated Nash equilibrium, and $\boldsymbol{q} \in \mathscr{S}$ be an optimal strategy profile.

Consider an arbitrary player $i \in [n]$. Since $\boldsymbol{P}$ is a Nash equilibrium, player $i$ cannot improve by switching to the pure strategy $q_i$. That is,

$$\begin{aligned}
\mathrm{PC}_i(\boldsymbol{P}) \leq \mathrm{PC}_i(\boldsymbol{P}_{-i}, q_i) &= \sum_{\boldsymbol{s}\in\mathscr{S}} (\boldsymbol{P}_{-i}, q_i)(\boldsymbol{s}) \sum_{e\in s_i} f_e(l_e(\boldsymbol{s})) \\
&= \sum_{\boldsymbol{s}\in\mathscr{S}} \boldsymbol{P}(\boldsymbol{s}) \sum_{e\in q_i} f_e(l_e(\boldsymbol{s}_{-i}, q_i)) \\
&\leq \sum_{\boldsymbol{s}\in\mathscr{S}} \boldsymbol{P}(\boldsymbol{s}) \sum_{e\in q_i} f_e(l_e(\boldsymbol{s}) + l_e(\boldsymbol{q}))\,.
\end{aligned} \tag{3.1}$$

Here, (3.1) follows because for all pure strategy profiles $\boldsymbol{s} \in \mathscr{S}$ and all $e \in q_i$ we have $l_e(\boldsymbol{s}_{-i}, q_i) \leq l_e(\boldsymbol{s}) + w_i \leq l_e(\boldsymbol{s}) + l_e(\boldsymbol{q})$. Summing up over all players $i \in [n]$ yields

$$\begin{aligned}
\mathrm{SC}(\boldsymbol{P}) = \sum_{i=1}^{n} w_i \cdot \mathrm{PC}_i(\boldsymbol{P}) &\leq \sum_{i=1}^{n} w_i \sum_{\boldsymbol{s}\in\mathscr{S}} \boldsymbol{P}(\boldsymbol{s}) \sum_{e\in q_i} f_e(l_e(\boldsymbol{s}) + l_e(\boldsymbol{q})) \\
&= \sum_{\boldsymbol{s}\in\mathscr{S}} \boldsymbol{P}(\boldsymbol{s}) \sum_{e\in E} l_e(\boldsymbol{q}) \cdot f_e(l_e(\boldsymbol{s}) + l_e(\boldsymbol{q}))\,.
\end{aligned}$$

Now, $l_e(\boldsymbol{q})$ and $l_e(\boldsymbol{s})$ are both non-negative reals. Thus, any pair $(c_1, c_2) \in (0,1) \times \mathbb{R}$ such that

$$\forall x,y \in \mathbb{R}_{\geq 0}, f \in \mathcal{P}_d : y \cdot f(x+y) \leq c_1 \cdot x \cdot f(x) + c_2 \cdot y \cdot f(y)$$

gives the bound

$$\begin{aligned}
\mathrm{SC}(\boldsymbol{P}) &\leq \sum_{\boldsymbol{s}\in\mathscr{S}} \boldsymbol{P}(\boldsymbol{s}) \sum_{e\in E} \left[c_1 \cdot l_e(\boldsymbol{s}) \cdot f_e(l_e(\boldsymbol{s})) + c_2 \cdot l_e(\boldsymbol{q}) \cdot f_e(l_e(\boldsymbol{q}))\right] \\
&= c_1 \cdot \mathrm{SC}(\boldsymbol{P}) + c_2 \cdot \mathrm{SC}(\boldsymbol{q})\,,
\end{aligned}$$

i.e.,

$$\frac{\mathrm{SC}(\boldsymbol{P})}{\mathrm{SC}(\boldsymbol{q})} \leq \frac{c_2}{1-c_1}\,.$$

Since $\Gamma$ and $\boldsymbol{P}$ were chosen arbitrarily, this implies

$$\mathrm{PoA}_{\mathrm{corr}}(\mathcal{U}(\mathcal{P}_d)) \leq \frac{c_2}{1-c_1}\,.$$

According to Theorem 1, $(c_1, c_2)$ can be chosen "optimally" in order to give the desired result. $\square$

## 3.2 Lower Bound

**Theorem 3.** *Let $d \in \mathbb{N}$. Then there is a weighted congestion game $\Gamma \in \mathcal{W}(\mathcal{P}_d)$ that has a pure Nash equilibrium $\boldsymbol{p} \in \mathscr{E}_{\mathrm{pure}}$ with*

$$\frac{\mathrm{SC}(\boldsymbol{p})}{\mathrm{OPT}} \geq \Phi_d^{d+1}\,.$$

*Proof.* Let $k \geq \max\{\binom{d}{\lfloor d/2 \rfloor}, 2\}$. Note that $\binom{d}{\lfloor d/2 \rfloor} = \max_{j \in [d]_0} \binom{d}{j}$. We construct a congestion game with $n = (d+1) \cdot k$ players and $|E| = n$ resources.

We divide the set $E$ into $d+1$ partitions: For $i \in [d]_0$, let $E_i := \{g_{i,1}, \ldots, g_{i,k}\}$, with each $g_{i,j}$ sharing the latency function $x \mapsto a_i \cdot x^d$. The values of the coefficients $a_i$ will be determined later. For simplicity of notation, set $g_{i,j} := g_{i,j-k}$ for $j > k$ in the following.

Similarly, we partition the set of players $[n]$: For $i \in [d]_0$, let $N_i := \{u_{i,1}, \ldots, u_{i,k}\}$. The weight of each player in set $N_i$ is $\Phi_d^i$, so $w_{u_{i,j}} = \Phi_d^i$ for all $i \in [d]_0$, $j \in [k]$.

Now, for every set $N_i$, each player $u_{i,j} \in N_i$ has exactly two strategies,

$$q_{u_{i,j}} := \{g_{i,j}\} \qquad \text{and} \qquad p_{u_{i,j}} := \begin{cases} \{g_{d,j+1}, \ldots, g_{d,j+\binom{d}{i}}, g_{i-1,j}\} & \text{if } i \in [d] \\ \{g_{d,j+1}\} & \text{if } i = 0\,. \end{cases}$$

Consider the pure strategy profiles $\boldsymbol{q} = (q_1, \ldots, q_n)$ and $\boldsymbol{p} = (p_1, \ldots, p_n)$. The resources in each set $E_i$ then have $\Phi_d$ times as much load under $\boldsymbol{p}$ as they have under $\boldsymbol{q}$:

| | **load on every resource** $e \in E_i$ | |
|---|---|---|
| $i$ | $l_e(\boldsymbol{q})$ | $l_e(\boldsymbol{p})$ |
| $d$ | $\Phi_d^d$ | $\sum_{l=0}^{d} \binom{d}{l} \Phi_d^l = (\Phi_d + 1)^d = \Phi_d^{d+1}$ |
| $0$ to $d-1$ | $\Phi_d^i$ | $\Phi_d^{i+1}$ |

For $\boldsymbol{p}$ to be a Nash equilibrium, we need to fulfill the following Nash conditions for each set $N_i$ of players:

| $i$ | **Nash condition to fulfill** |
|---|---|
| $1$ to $d$ | $\mathrm{PC}_{u_{i,j}}(\boldsymbol{p}) = \binom{d}{i} \cdot a_d \cdot (\Phi_d^{d+1})^d + a_{i-1} \cdot (\Phi_d^i)^d$ $\leq a_i \cdot (\Phi_d^{i+1} + \Phi_d^i)^d = \mathrm{PC}_{u_{i,j}}(\boldsymbol{p}_{-u_{i,j}}, q_{u_{i,j}})$ |
| $0$ | $\mathrm{PC}_{u_{0,j}}(\boldsymbol{p}) = a_d \cdot (\Phi_d^{d+1})^d \leq a_0 \cdot (\Phi_d + 1)^d = \mathrm{PC}_{u_{0,j}}(\boldsymbol{p}_{-u_{0,j}}, q_{u_{0,j}})$ |

Replacing "$\leq$" by "$=$" yields a homogeneous system of linear equations, that is, the system $B_d \cdot a = 0$ where $B_d$ is the following $(d+1) \times (d+1)$ matrix:

$$B_d = \begin{pmatrix} -\Phi_d^{d^2+d+1} + \Phi_d^{d^2+d} & \Phi_d^{d^2} & 0 & \cdots & & & & & \cdots & 0 \\ \binom{d}{d-1}\Phi_d^{d^2+d} & -\Phi_d^{d^2+1} & \ddots & & & & & & & \vdots \\ \vdots & 0 & \ddots & & & & & & & \\ \vdots & \vdots & & \ddots & \ddots & & & & & \vdots \\ \binom{d}{i}\Phi_d^{d^2+d} & 0 & \cdots & 0 & -\Phi_d^{id+d+1} & \Phi_d^{id} & 0 & \cdots & & 0 \\ \vdots & \vdots & & & 0 & \ddots & \ddots & & & \vdots \\ & & & & \vdots & & & \ddots & & 0 \\ \vdots & \vdots & & & \vdots & & & \ddots & & \Phi_d^d \\ \Phi_d^{d^2+d} & 0 & \cdots & & 0 & \cdots & & 0 & & -\Phi_d^{d+1} \end{pmatrix} \tag{3.2}$$

and $a := (a_d \dots a_0)^t$. Obviously, a solution to this system fulfills the initial Nash conditions. Note that

$$(\Phi_d^{i+1} + \Phi_d^i)^d = (\Phi_d^i)^d \cdot (\Phi_d + 1)^d = \Phi_d^{id+d+1}.$$

We proceed by showing a property of $B_d$.

*Claim.* The $(d+1) \times (d+1)$ matrix $B_d$ from (3.2) has rank $d$.

*Proof (of the claim).* Consider the matrix $C_d$ that results from adding row $j$ multiplied by the factor $\Phi_d^{-1}$ to row $j-1$, sequentially done for $j = d+1, d, \dots, 2$. Obviously, $C_d$ is a lower triangular matrix with nonzero elements only in the first column and on the principal diagonal.

For the top left element of $C_d$ we get

$$-\Phi_d^{d^2+d+1} + \sum_{j=0}^{d} \binom{d}{j} \Phi_d^{d^2+j} = \Phi_d^{d^2} \cdot \Bigg( -\Phi_d^{d+1} + \underbrace{\sum_{j=0}^{d} \binom{d}{j} \Phi_d^j}_{(\Phi_d+1)^d} \Bigg) = 0.$$

Since all elements on the principal diagonal of $C_d$—with the just shown exception of the first one—are nonzero, it is easy to see that $C_d$ (and thus also $B_d$) has rank $d$. ∎

By the above claim it follows that the column vectors of $B_d$ are linearly dependent and thus there are—with degree of freedom 1—infinitely many linear combinations of them yielding 0. In other words, $B_d \cdot a = 0$ has a one-dimensional solution space.

We now show (by induction over $i$) that all coefficients $a_i$, $i \in [d]_0$ must have the same sign and thus we can always find a valid solution. From the last equality, for $i = 0$, we have that $a_d$ and $a_0$ must have the same sign. Now for $i = 1, \dots, d-1$, it follows that $a_i$ must have the same sign as $a_{i-1}$ and $a_d$, for $(\Phi_d^{d+1})^d$, $(\Phi_d^i)^d$, and $(\Phi_d^{i+1} + \Phi_d^i)^d$ are all positive.

Choosing $a \neq 0$ with all components being positive, all coefficients of the latency functions are positive. We get

$$\frac{\mathrm{SC}(\boldsymbol{p})}{\mathrm{OPT}} \geq \frac{\mathrm{SC}(\boldsymbol{p})}{\mathrm{SC}(\boldsymbol{q})} = \frac{k \cdot \sum_{i=0}^{d} a_i (\Phi_d^{i+1})^{d+1}}{k \cdot \sum_{i=0}^{d} a_i (\Phi_d^{i})^{d+1}} = \Phi_d^{d+1}. \qquad \square$$

**Theorem 4.** *The lower bound on the pure price of anarchy from Theorem 3 also holds for (directed) network congestion games.*

*Proof.* Each instance of the congestion game in Theorem 3 can be characterized by two parameters: The maximum degree $d$ of the latency functions and the number of resources $k \geq \max\{\binom{d}{\lfloor d/2 \rfloor}, 2\}$ in each class $E_i$, where $i \in [d]_0$. Recall that the number of resources—as well as the number of players—is given by $(d+1) \cdot k$. Given such an instance we construct a network congestion game without altering the lower bound on the pure price of anarchy ($\mathrm{PoA}_{\mathrm{pure}}$).

Confer Figure 1 for an example in the case of quadratic latency functions (i.e., $d = 2$) and $k = 2$. For their respective players, gray nodes denote origins, whereas nodes with a thick outline represent destinations. Note that for the sake of clarity not all edges are shown, as will be explained later. Edges without a label have $f_e(x) = 0$ as their latency function. We call these edges *free* edges. All other edges have the associated latency function as in Theorem 3. In the following, we will outline the general construction scheme.

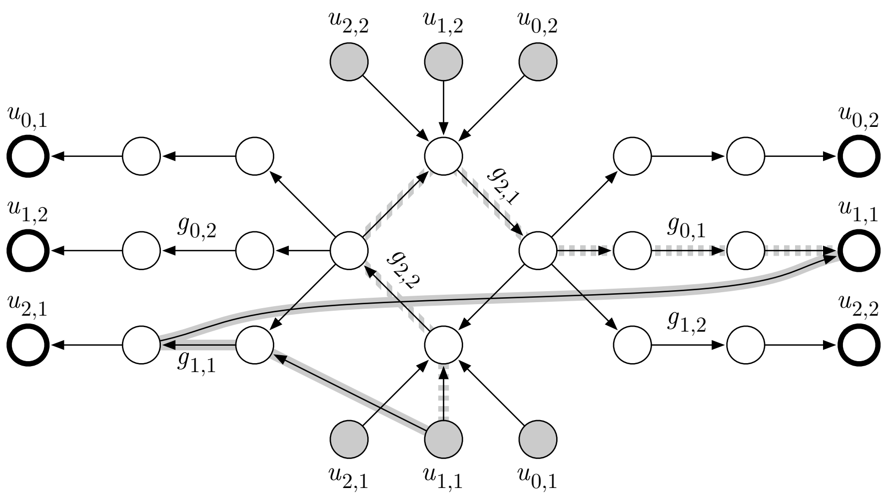


**Figure 1:** Weighted network congestion game for $d = 2$ and $k = 2$

The network corresponding to an instance $(d, k)$ can be constructed as follows: There is a circle of $2 \cdot k$ edges where every other edge represents $g_{d,1}, g_{d,2}, \ldots, g_{d,k}$. All remaining edges in the circle are free edges. Furthermore, every player $u_{i,j}$ has its own origin node which has a single free edge to $g_{d,j+1}$. Consequently, circle edge $g_{d,j+\binom{d}{i}}$ connects to a free edge which then in turn connects to edge $g_{i-1,j}$. (In case $i = 0$, the latter is simply another free edge.) From there, another free edge to the destination node of player $u_{i,j}$ exists. Note that, thus far, the graph has exactly one acyclic path for each player, that is, for each origin-destination pair. Each of these paths represents the respective player's "unfavorable" strategy which has been denoted as $P_{u_{i,j}}$ in Theorem 3.

We can now add two more free edges for each player $u_{i,j}$ that allow him to also use his optimal strategy $Q_{u_{i,j}}$: From $u_{i,j}$'s origin node add a free link to $g_{i,j}$, and from $g_{i,j}$ add a free link to $u_{i,j}$'s destination node. We call the first type of links *A-links*, the latter *B-links*. Note that in Figure 1, A- and B-links are only shown for player $u_{1,1}$. (The figure is complete otherwise.) The thick gray path denotes player $u_{1,1}$'s strategy in the system optimum, whereas the hatched path indicates his strategy in the worst-case Nash equilibrium.

A-links obviously cannot create shortcuts for *other* players as origin nodes only have outgoing edges. Similarly, destination nodes only have incoming edges and therefore B-links cannot create shortcuts for other players, either. Eventually, neither A- nor B-links can create a shortcut for the same player's other strategy $P_{u_{*,*}}$ as they do not share any nodes, except for the origin and destination nodes.

Note, however, that B-links do create additional paths: In Figure 1, for instance, player $u_{1,1}$ now has the further option of using a path consisting of five edges: three free ones, $g_{2,2}$, and $g_{1,1}$. Nevertheless, all such additional paths are supersets of the player's optimal strategy and thus neither change the system optimum nor the worst-case Nash equilibrium. □

# 4 Unweighted Congestion Games

In this section, we establish the exact price of anarchy for unweighted congestion games with polynomial latency function of degree at most $d$. The structure of this section is very similar to the previous Section 3, yet there are some subtle differences when players are unweighted. In particular, the price of anarchy turns out to be slightly smaller than for the weighted case.

## 4.1 Upper Bound

Again, the upper bound crucially relies on a purely analytical result. We defer its proof to Section 5.4.

**Theorem 5.** *Let $d \in \mathbb{N}$ and define $k := \lfloor \Phi_d \rfloor$. Then*

$$\min_{(c_1,c_2)\in(0,1)\times\mathbb{R}} \left\{ \frac{c_2}{1-c_1} \;\middle|\; \forall x,y \in \mathbb{N}_0, f \in \mathcal{P}_d : y \cdot f(x+1) \le c_1 \cdot x \cdot f(x) + c_2 \cdot y \cdot f(y) \right\}$$
$$= \frac{(k+1)^{2d+1} - k^{d+1}(k+2)^d}{(k+1)^{d+1} - (k+2)^d + (k+1)^d - k^{d+1}}\,.$$

We remark that in Corollary 3 of Section 5.4, we also give concrete values for $c_1$ and $c_2$ so that the minimum is attained. In particular, this shows that the minimum on the left-hand side of Theorem 5 exists and is well-defined.

**Theorem 6.** *Let $d \in \mathbb{N}$ and define $k := \lfloor \Phi_d \rfloor$. Then,*

$$\mathrm{PoA}_{\mathrm{corr}}(\mathcal{U}(\mathcal{P}_d)) \le \frac{(k+1)^{2d+1} - k^{d+1}(k+2)^d}{(k+1)^{d+1} - (k+2)^d + (k+1)^d - k^{d+1}}\,.$$

*Proof.* The proof is very similar to the proof of Theorem 2. Let $\Gamma$ be an arbitrary unweighted congestion game, $\boldsymbol{P} \in \Delta(\mathscr{S})$ be a correlated Nash equilibrium, and $\boldsymbol{q} \in \mathscr{S}$ be an optimal strategy profile.

Consider an arbitrary player $i \in [n]$. Since $\boldsymbol{P}$ is a Nash equilibrium, player $i$ cannot improve by switching to the pure strategy $q_i$. That is,

$$\begin{aligned}
\mathrm{PC}_i(\boldsymbol{P}) \leq \mathrm{PC}_i(\boldsymbol{P}_{-i}, q_i) &= \sum_{\boldsymbol{s} \in \mathscr{S}} (\boldsymbol{P}_{-i}, q_i)(\boldsymbol{s}) \sum_{e \in s_i} f_e(l_e(\boldsymbol{s})) \\
&= \sum_{\boldsymbol{s} \in \mathscr{S}} \boldsymbol{P}(\boldsymbol{s}) \sum_{e \in q_i} f_e(l_e(\boldsymbol{s}_{-i}, q_i)) \\
&\leq \sum_{\boldsymbol{s} \in \mathscr{S}} \boldsymbol{P}(\boldsymbol{s}) \sum_{e \in q_i} f_e(l_e(\boldsymbol{s}) + 1)\,. \qquad (4.1)
\end{aligned}$$

Here, (4.1) follows because for all pure strategy profiles $\boldsymbol{s} \in \mathscr{S}$ and all $e \in q_i$ we have $l_e(\boldsymbol{s}_{-i}, q_i) \leq l_e(\boldsymbol{s}) + 1$. Equality holds if and only if $e \notin s_i$. Summing up over all players $i \in [n]$ yields

$$\mathrm{SC}(\boldsymbol{P}) = \sum_{i=1}^{n} \mathrm{PC}_i(\boldsymbol{P}) \leq \sum_{i=1}^{n} \sum_{\boldsymbol{s} \in \mathscr{S}} \boldsymbol{P}(\boldsymbol{s}) \sum_{e \in q_i} f_e(l_e(\boldsymbol{s}) + 1) = \sum_{\boldsymbol{s} \in \mathscr{S}} \boldsymbol{P}(\boldsymbol{s}) \sum_{e \in E} l_e(\boldsymbol{q}) \cdot f_e(l_e(\boldsymbol{s}) + 1)\,.$$

Now, $l_e(\boldsymbol{q})$ and $l_e(\boldsymbol{s})$ are both integers, since $\boldsymbol{q}$ and $\boldsymbol{s}$ are both pure strategy profiles. Thus, any pair $(c_1, c_2) \in (0,1) \times \mathbb{R}$ such that

$$\forall x, y \in \mathbb{N}_0, f \in \mathcal{P}_d : y \cdot f(x+1) \leq c_1 \cdot x \cdot f(x) + c_2 \cdot y \cdot f(y)$$

gives the bound

$$\begin{aligned}
\mathrm{SC}(\boldsymbol{P}) &\leq \sum_{\boldsymbol{s} \in \mathscr{S}} \boldsymbol{P}(\boldsymbol{s}) \sum_{e \in E} [c_1 \cdot l_e(\boldsymbol{s}) \cdot f_e(l_e(\boldsymbol{s})) + c_2 \cdot l_e(\boldsymbol{q}) \cdot f_e(l_e(\boldsymbol{q}))] \\
&= c_1 \cdot \mathrm{SC}(\boldsymbol{P}) + c_2 \cdot \mathrm{SC}(\boldsymbol{q})
\end{aligned}$$

i.e.,

$$\frac{\mathrm{SC}(\boldsymbol{P})}{\mathrm{SC}(\boldsymbol{q})} \leq \frac{c_2}{1 - c_1}\,.$$

Since $\Gamma$ and $\boldsymbol{P}$ were chosen arbitrarily, this implies

$$\mathrm{PoA}_{\mathrm{corr}}(\mathcal{U}(\mathcal{P}_d)) \leq \frac{c_2}{1 - c_1}\,.$$

According to Theorem 5, $(c_1, c_2)$ can be chosen "optimally" in order to give the desired result. $\square$

### 4.2 Lower Bound

**Theorem 7.** *Let $d \in \mathbb{N}$. Then there is an unweighted congestion game $\Gamma \in \mathcal{U}(\mathcal{P}_d)$ that has a pure Nash equilibrium $\boldsymbol{p} \in \mathscr{E}_{\mathrm{pure}}$ with*

$$\frac{\mathrm{SC}(\boldsymbol{p})}{\mathrm{OPT}} \geq \frac{(k+1)^{2d+1} - k^{d+1}(k+2)^d}{(k+1)^{d+1} - (k+2)^d + (k+1)^d - k^{d+1}}\,.$$

*Proof.* Let $k := \lfloor \Phi_d \rfloor$. We construct a congestion game with $n \geq k+2$ players and $|E| = 2 \cdot n$ resources.

We divide the set $E$ into two subsets $E_1 := \{g_1, \ldots, g_n\}$ and $E_2 := \{h_1, \ldots, h_n\}$. Each player $i \in [n]$ has two pure strategies, $q_i := \{g_i, h_i\}$ and $p_i := \{g_{i+1}, \ldots, g_{i+k}, h_{i+1}, \ldots, h_{i+k+1}\}$ where we denote $g_j := g_{j-n}$ and $h_j := h_{j-n}$ for $j > n$. The strategy set of player $i$ is thus $\mathscr{S}_i := \{q_i, p_i\}$.

Each of the resources in $E_1$ share the latency function $x \mapsto ax^d$ for an $a \in \mathbb{R}_{>0}$ yet to be determined, whereas the resources in $E_2$ have latency $x \mapsto x^d$.

Obviously, the optimal allocation is $\boldsymbol{q} = (q_1, \ldots, q_n)$ where every player $i \in [n]$ chooses strategy $q_i$. Now we determine a value for $a$ such that the allocation $\boldsymbol{p} = (p_1, \ldots, p_n)$ is a worst-case Nash equilibrium. By definition, we must hold for all players $i \in [n]$ that $\mathrm{PC}_i(\boldsymbol{p}) \leq \mathrm{PC}_i(\boldsymbol{p}_{-i}, q_i)$, or equivalently

$$k \cdot a \cdot k^d + (k+1) \cdot (k+1)^d \leq a \cdot (k+1)^d + (k+2)^d\,.$$

Solving for $a$ gives

$$a \geq \frac{(k+1)^{d+1} - (k+2)^d}{(k+1)^d - k^{d+1}} > 0\,. \tag{4.2}$$

Note that $a$ is positive due to the definition of $k$. Since for any player $i \in [n]$ the private costs are $\mathrm{PC}_i(\boldsymbol{q}) = a + 1$ and $\mathrm{PC}_i(\boldsymbol{p}) = a \cdot k^{d+1} + (k+1)^{d+1}$, it follows that

$$\frac{\mathrm{SC}(\boldsymbol{p})}{\mathrm{OPT}} \geq \frac{\mathrm{SC}(\boldsymbol{p})}{\mathrm{SC}(\boldsymbol{q})} = \frac{\sum_{i=1}^{n} \mathrm{PC}_i(\boldsymbol{p})}{\sum_{i=1}^{n} \mathrm{PC}_i(\boldsymbol{q})} = \frac{a \cdot k^{d+1} + (k+1)^{d+1}}{a+1}\,. \tag{4.3}$$

Since $(k+1)^d \geq k^{d+1}$, it is not hard to see that the right-hand side of (4.3) is monotonically decreasing in $a$. Thus, we assume equality in (4.2), which then gives the desired result. □

**Theorem 8.** *The lower bound on the pure price of anarchy from Theorem 7 also holds for (directed) network congestion games.*

*Proof.* Instances of the congestion games in Theorem 7 can be characterized by two parameters: The maximum degree $d$ of the latency functions and the number of players $n \geq k+2$ where $k := \lfloor \Phi_d \rfloor$. The number of resources is then $2n$.

Figure 2 (left) shows an example of the network congestion game for quadratic latency functions (i.e., $d = 2$) and for $n = 4$ players. Unlabeled edges $e \in E$ have $f_e(x) = 0$ as their latency function. As in the proof of Theorem 4, we say that these edges are *free*. All other edges have the associated latency function as in Theorem 6. In the following we outline the construction scheme.

The network corresponding to an instance characterized by $(d, n)$ can be constructed as follows: There is a circle of $2n$ undirected edges $g_1, h_1, g_2, h_2, \ldots, g_n, h_n$. Each undirected edge $(v_1, v_2)$ has to be replaced by the construction shown in Figure 2 (right). This ensures that no matter in which direction a player uses edge $(v_1, v_2)$ it produces load on the directed edge $(v_3, v_4)$.

Now, every player $i$ has its own *origin node* outside the circle—which is indicated by a gray background in the example. This origin node is connected with a free edge to the node incident to $g_i$ and $h_{i-1}$. The *destination node* of each player $i$ is the node incident to $g_{i+1}$ and $h_i$, represented by a thick outline in the figure. To also allow each player $i$ to choose strategy $p_i$ (which essentially goes the other way round inside the circle), we finally add one more free edge from $i$'s origin node to the connecting node of $g_{i+k+1}$ and $h_{i+k+1}$. □

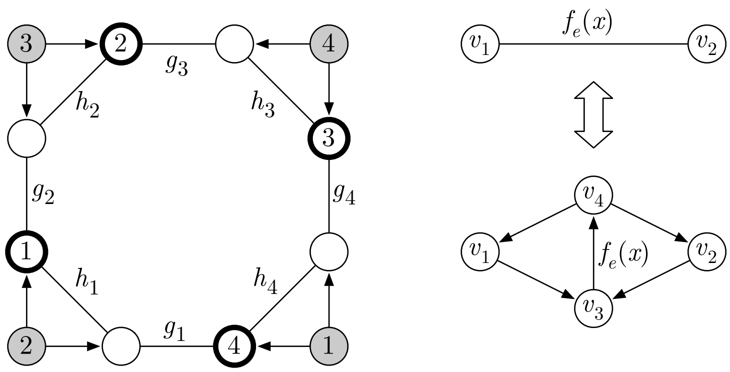


**Figure 2:** Network congestion game for $d = 2$ and 4 players

### 4.3 Comparison to Weighted Games

As a corollary of Theorems 1 and 5, the price of anarchy for unweighted games is not "too far" away from the price of anarchy for weighted games.

**Corollary 1.** *Let $d \in \mathbb{N}$ and define $k := \lfloor \Phi_d \rfloor$. Then,*

$$\lfloor \Phi_d \rfloor^{d+1} \leq \frac{(k+1)^{2d+1} - k^{d+1}(k+2)^d}{(k+1)^{d+1} - (k+2)^d + (k+1)^d - k^{d+1}} \leq \Phi_d^{d+1}.$$

*Proof.* The upper bound is a direct consequence of Theorems 1 and 5. For the lower bound, define $A := (k+1)^d - k^{d+1}$ and $B := (k+1)^{d+1} - (k+2)^d$. Note that $A, B > 0$ by choice of $k$. Then

$$\begin{aligned}\frac{(k+1)^{2d+1} - k^{d+1}(k+2)^d}{(k+1)^{d+1} - (k+2)^d + (k+1)^d - k^{d+1}} &= \frac{(A + k^{d+1})(k+1)^{d+1} - k^{d+1}(k+2)^d}{A+B} \\ &= k^{d+1} \cdot \frac{A \cdot \left(\frac{k+1}{k}\right)^{d+1} + B}{A+B} \geq \lfloor \Phi_d \rfloor^{d+1}.\end{aligned}$$ □

## 5 Analytical Tools

In this section, we prove the irrationality of $\Phi_d$, establish the asymptotic behavior of $\Phi_d$, and give the proofs of Theorems 1 and 5. These results are based on several technical lemmas that we state and prove in advance.

### 5.1 Preliminaries

**Lemma 1.** *Let $c \in (0, 1]$ and $x \in \mathbb{R}_{\geq 0}$. Define $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $g(r) := (x+1)^r - c \cdot x^{r+1}$. Now it holds for all $d, r \in \mathbb{R}_{\geq 0}$ with $d > r$ and $g(r) \geq 0$ that $g(d) \geq g(r)$.*

*Proof.* Let $d, r \in \mathbb{R}_{\geq 0}$ with $d > r$ and $g(r) \geq 0$. Since for $x = 0$ we have $g(d) = g(r) = 1$, we only consider the case $x > 0$ in the following. We get

$$\begin{aligned} g'(r) &= (x+1)^r \cdot \ln(x+1) - c \cdot x^{r+1} \cdot \ln(x) \\ &> \ln(x+1) \left[ (x+1)^r - c \cdot x^{r+1} \right] \geq 0 \,. \end{aligned}$$

By way of contradiction, assume $g(d) \leq g(r)$ and let $\xi \in (r, d)$ be minimal with $g'(\xi) = 0$. Such a $\xi$ must exist due to the intermediate and mean value theorems. By definition of $\xi$, however, we have $g(\xi) > g(r) = 0$. Hence, $g'(\xi) > 0$. A contradiction. $\square$

**Lemma 2.** *Let $c \in (0, 1]$ and $d \in \mathbb{N}$. Define $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $g(x) := (x+1)^d - c \cdot x^{d+1}$. Then, it holds that $g$ has exactly one local maximum, at some $\xi \in \mathbb{R}_{>0}$. Moreover, $g$ is strictly increasing in $[0, \xi)$, and strictly decreasing in $(\xi, \infty)$.*

*Proof.* We have as necessary first-order condition for a local extremum that

$$g'(x) = d(x+1)^{d-1} - c(d+1)x^d = 0 \,, \quad \text{i.e.,} \quad \left[ x > 0 \text{ and } \frac{(x+1)^{d-1}}{x^d} = \frac{c \cdot (d+1)}{d} \right] . \tag{5.1}$$

Now, $\lim_{x \downarrow 0} \frac{(x+1)^{d-1}}{x^d} = \infty$, $\lim_{x \to \infty} \frac{(x+1)^{d-1}}{x^d} = 0$, and for all $x \in \mathbb{R}_{>0}$ it holds that

$$\frac{\partial}{\partial x} \frac{(x+1)^{d-1}}{x^d} = - \frac{(x+1)^{d-2} \cdot (x+d)}{x^{d+1}} < 0 \,.$$

As a result, there is a unique $\xi \in \mathbb{R}_{>0}$ with $\frac{(\xi+1)^{d-1}}{\xi^d} = \frac{c \cdot (d+1)}{d}$. Since $\lim_{x \to \infty} g(x) = -\infty$, it follows that $g$ has both a local and a global maximum at $\xi$ and that $g$ is strictly increasing in $[0, \xi)$ and strictly decreasing in $(\xi, \infty)$. $\square$

## 5.2 Properties of $\Phi_d$

**Theorem 9.** *For all $d \in \mathbb{N}$ it holds that $\Phi_d$ is irrational.*

*Proof (by contradiction).* Let $d \in \mathbb{N}$. From Lemmata 1 and 2 it follows that $\Phi_d$ is well-defined and $\Phi_d > 1$. Assume now that $\Phi_d$ is rational, i.e., there are coprime $a, b \in \mathbb{N}$ with

$$\Phi_d = \frac{a}{b} \,.$$

Dividing $(\Phi_d + 1)^d = \Phi_d^{d+1}$ by $\Phi_d^d$ gives

$$\Phi_d = \left( 1 + \frac{1}{\Phi_d} \right)^d \,.$$

Hence,

$$\frac{a}{b} = \frac{(a+b)^d}{a^d} \,.$$

Since $a$ and $b$ are coprime, so are $a^d$ and $b$. Moreover, $\frac{a}{b}$ is in reduced form, and thus $b$ is a divisor of $a^d$. Consequently, it must hold that $a = b = 1$. This is a contradiction to $\Phi_d > 1$. $\square$

**Theorem 10.** $\Phi_d = \Theta(\frac{d}{\ln d})$.

*Proof.* We show that there is some $d_0$ so that for all $d \geq d_0$ it holds that $\frac{2d}{\ln d} \geq \Phi_d$ and $\frac{d}{2\ln d} < \Phi_d$. Let $c > 0$. Due to Lemma 2, it holds that $\frac{c \cdot d}{\ln d} \geq \Phi_d$ if and only if

$$\left(\frac{c \cdot d}{\ln d} + 1\right)^d < \left(\frac{c \cdot d}{\ln d}\right)^{d+1} .$$

Taking the $d$-th root, multiplying with $\frac{\ln d}{c \cdot d}$, and raising to the $d$-th power again yields the equivalent statement

$$\left(1 + \frac{\ln d}{c \cdot d}\right)^d < \frac{c \cdot d}{\ln d} .$$

Now recall that the tailor series expansion of $f(x) := (1+x)^n$ about 0 is

$$f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!} \cdot x^i = \sum_{i=0}^{n} \frac{n! \cdot x^i}{(n-i)! \cdot i!} ,$$

where $f^{(i)}$ denotes the $i$-th derivative of $f$. Hence,

$$\left(1 + \frac{\ln d}{c \cdot d}\right)^d = \sum_{i=0}^{d} \frac{d!}{d^i \cdot (d-i)!} \cdot \frac{\left(\frac{\ln d}{c}\right)^i}{i!} .$$

Consequently, when $c = 2$,

$$\left(1 + \frac{\ln d}{2 \cdot d}\right)^d < \sum_{i=0}^{d} \frac{\left(\frac{\ln d}{2}\right)^i}{i!} < \sum_{i=0}^{\infty} \frac{\left(\frac{\ln d}{2}\right)^i}{i!} = \exp\left(\frac{\ln d}{2}\right) = \sqrt[2]{d} < \frac{2 \cdot d}{\ln d}$$

for all $d \geq 2$. This proves the upper bound $\frac{2d}{\ln d} \geq \Phi_d$.

When $c = \frac{1}{2}$, define $n(d) := \lfloor \frac{d}{2} \rfloor$. Then,

$$\left(1 + \frac{\ln d}{2 \cdot d}\right)^d = \sum_{i=0}^{d} \frac{2^i \cdot d!}{d^i \cdot (d-i)!} \cdot \frac{\ln^i d}{i!} > \sum_{i=0}^{n(d)} \frac{2^i \cdot d!}{d^i \cdot (d-i)!} \cdot \frac{\ln^i d}{i!} > \sum_{i=0}^{n(d)} \frac{\ln^i d}{i!} = d - R_{n(d)+1}(\ln d) ,$$

where

$$R_{n(d)+1}(\ln d) \leq 2 \cdot \frac{\ln^{n(d)+1} d}{(n(d)+1)!} \to 0 \text{ for } d \to \infty .$$

(See [10], for instance.) Hence, when $d$ is sufficiently large, then

$$\left(1 + \frac{\ln d}{2 \cdot d}\right)^d > \frac{d}{2\ln d} .$$

This proves the lower bound $\frac{d}{2\ln d} < \Phi_d$. Numerically, it can even be verified that this holds for all $d \geq 2$. $\square$

### 5.3 Weighted Games: Proof of Theorem 1

As a first step of determining the exact value of Theorem 1, we simplify the expression to get rid of quantification over all polynomials.

**Lemma 3.** *Let $d \in \mathbb{N}$. Then,*

$$I := \inf_{(c_1,c_2)\in(0,1)\times\mathbb{R}} \left\{ \frac{c_2}{1-c_1} \,\middle|\, \forall x, y \in \mathbb{R}_{\geq 0}, f \in \mathcal{P}_d : y \cdot f(x+y) \leq c_1 \cdot x \cdot f(x) + c_2 \cdot y \cdot f(y) \right\}$$
$$= \inf_{c\in(0,1)} \left\{ \max_{x\in\mathbb{R}_{\geq 0}} \left\{ \frac{(x+1)^d - c \cdot x^{d+1}}{1-c} \right\} \right\}$$

*Proof.* We first note that for any $c \in (0,1)$ the maximum on the right-hand side exists (Lemma 2).
Assume there is a pair $(c_1, c_2) \in (0,1) \times \mathbb{R}$ with

$$\forall x, y \in \mathbb{R}_{\geq 0}, f \in \mathcal{P}_d : y \cdot f(x+y) \leq c_1 \cdot x \cdot f(x) + c_2 \cdot y \cdot f(y)\,. \tag{5.2}$$

Since each $f \in \mathcal{P}_d$ is a linear combination of monomials, (5.2) may be simplified such that $f$ ranges only over all monomials of maximum degree $d$. Moreover, when $y = 0$ the condition is fulfilled trivially. Therefore, (5.2) is equivalent to

$$\forall x \in \mathbb{R}_{\geq 0}, y \in \mathbb{R}_{>0}, r \in [d]_0 : y \cdot (x+y)^r \leq c_1 \cdot x^{r+1} + c_2 \cdot y^{r+1}\,,$$

which is the same as

$$\forall x \in \mathbb{R}_{\geq 0}, y \in \mathbb{R}_{>0}, r \in [d]_0 : c_2 \geq \left(\frac{x}{y} + 1\right)^r - c_1 \cdot \left(\frac{x}{y}\right)^{r+1}\,. \tag{5.3}$$

Clearly, this is equivalent to

$$\forall x \in \mathbb{R}_{\geq 0}, r \in [d]_0 : c_2 \geq (x+1)^r - c_1 \cdot x^{r+1}\,. \tag{5.4}$$

When $x = 0$, we have for all $r \in [d]_0$ that $(x+1)^r - c_1 \cdot x^{r+1} = 1 > 0$. Therefore, using Lemma 1, we can again simplify (5.4) to

$$\forall x \in \mathbb{R}_{\geq 0} : c_2 \geq (x+1)^d - c_1 \cdot x^{d+1}\,. \tag{5.5}$$

Finally, we get that

$$I = \inf_{(c_1,c_2)\in(0,1)\times\mathbb{R}} \left\{ \frac{c_2}{1-c_1} \,\middle|\, \forall x \in \mathbb{R}_{\geq 0} : c_2 \geq (x+1)^d - c_1 \cdot x^{d+1} \right\}$$
$$= \inf_{c\in(0,1)} \left\{ \max_{x\in\mathbb{R}_{\geq 0}} \left\{ \frac{(x+1)^d - c \cdot x^{d+1}}{1-c} \right\} \right\}. \qquad \square$$

Now, a straightforward way of evaluating the right-hand side of Lemma 3 would be to determine the value of the max-term with respect to any arbitrary $c \in (0,1)$. According to Lemma 2, the maximum is even unique. Hence, there is temptation to treat the max-term as a function of $c$ and then use ordinary calculus for computing the infimum $I$.

Unfortunately, this naive approach turns out to be infeasible as evaluating the max-term using first-order conditions requires solving for the roots of $x \mapsto (x+1)^d - c \cdot x^{d+1}$, i.e., a polynomial of degree $d-1$. Our proof is therefore based on the following observation, which exploits the first-order condition (5.1) of Lemma 2: For each $x \in \mathbb{R}_{\geq 0}$, the condition yields a unique $c \in (0,1)$ so that $x$ becomes a maximizer (on $\mathbb{R}_{\geq 0}$) of the aforementioned polynomial. Consequently, inserting into the right-hand side of Lemma 3 immediately gives an upper bound on the infimum $I$. In the following, we show that choosing $x = \Phi_d$ (and $c$ accordingly) makes the upper bound and $I$ coincide.

**Lemma 4.** *Let $d \in \mathbb{N}$. Then,*

$$I = \inf_{c \in (0,1)} \left\{ \max_{x \in \mathbb{R}_{\geq 0}} \left\{ \frac{(x+1)^d - c \cdot x^{d+1}}{1-c} \right\} \right\} = \Phi_d^{d+1} .$$

*Proof.* Define $g : (0,1) \times \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R}_{>0} \to \mathbb{R}$,

$$g(c,x) := \frac{(x+1)^d - c \cdot x^{d+1}}{1-c} \quad \text{and} \quad h(x) := \frac{d \cdot (x+1)^{d-1}}{(d+1) \cdot x^d} .$$

The proof proceeds in several steps. Clearly, $g$ is differentiable on $(0,1) \times \mathbb{R}$.

- Assume there is a pair $(\gamma, \xi) \in (0,1) \times \mathbb{R}_{>0}$ with $\gamma = h(\xi)$. Then $g(\gamma,\xi) = \max_{x \in \mathbb{R}_{\geq 0}} \{g(\gamma, x)\}$.

  Clearly, $\xi$ fulfills the necessary first order condition (5.1) for local extrema of $g(\gamma, \cdot)$. By Lemma 2, we get that $\xi$ is in fact the unique positive local maximum and that $g(\gamma, \cdot)$ is strictly increasing in $(0, \xi)$ and strictly decreasing in $(\xi, \infty)$. Clearly, this implies $g(\gamma, \xi) = \max_{x \in \mathbb{R}_{\geq 0}} \{g(\gamma, x)\}$.

- $c_1 := h(\Phi_d) \in (0,1)$

  Since $(\Phi_d + 1)^d = \Phi_d^{d+1}$, it holds that

$$c_1 = h(\Phi_d) = \frac{d \cdot (\Phi_d + 1)^d}{(d+1) \cdot \Phi_d^d \cdot (\Phi_d + 1)} = \frac{d \cdot \Phi_d}{(d+1) \cdot (\Phi_d + 1)} \in (0,1) . \tag{5.6}$$

Up to this point, we have shown that

$$I \leq \max_{x \in \mathbb{R}_{\geq 0}} \{g(c_1, x)\} = g(c_1, \Phi_d) = \frac{(\Phi_d + 1)^d - c_1 \cdot \Phi_d^{d+1}}{1 - c_1} = \Phi_d^{d+1} .$$

In the remainder of the proof we show that also "$\geq$" holds, thereby giving an a posteriori explanation why $c_1$ was chosen "optimally".

- $I \geq g(c_1, \Phi_d)$

  Clearly, for all $c \in (0,1)$, we have $g(c, \Phi_d) = \Phi_d^{d+1}$. That is, it holds for all $c \in (0,1) \setminus \{c_1\}$ that $\sup_{x \in \mathbb{R}_{\geq 0}} \{g(c,x)\} \geq g(c_1, \Phi_d)$. See Figure 3 for an illustration. □

**Corollary 2.** *The minimum in Theorem 5 is attained for*

$$c_1 = \frac{d \cdot \Phi_d}{(d+1) \cdot (\Phi_d + 1)} \quad \textit{and} \quad c_2 = \frac{(d+1+\Phi_d) \cdot (\Phi_d + 1)^{d-1}}{d+1}\,.$$

*Proof.* From the proof of Lemma 3, we know that for any fixed $c_1$ there is an "optimal" $c_2 = \max_{x \in \mathbb{R}_{\geq 0}}\{(x+1)^d - c_1 \cdot x^{d+1}\}$, see (5.5). From the proof of Lemma 4 we get that there is an "optimal" $c_1 \in (0,1)$, see (5.6). In particular, the infima in the previous lemmas are also minima. Inserting gives the desired result. ◻

## 5.4 Unweighted Games: Proof of Theorem 5

The proof of Theorem 5 is similar to the proof of Theorem 1. However, special care is necessary to account for the discrete variables. We start with a technical lemma that will allow us to make a simplification corresponding to the one from (5.3) to (5.4) in Lemma 3.

**Lemma 5.** *Let $d \in \mathbb{N}_0$. Then it holds for all $c \in (0,1)$ that*

$$\max_{x \in \mathbb{N}_0, y \in \mathbb{N}} \left\{ \left(\frac{x+1}{y}\right)^d - c \cdot \left(\frac{x}{y}\right)^{d+1} \right\} = \max_{x \in \mathbb{N}_0} \left\{ (x+1)^d - c \cdot x^{d+1} \right\}\,.$$

*Proof.* We first note that the maximum on the right-hand side exists (Lemma 2).

Define $g : \mathbb{N}_0 \times \mathbb{N} \times [0,1] \to \mathbb{R}$,

$$g(x,y,c) := \left(\frac{x+1}{y}\right)^d - c \cdot \left(\frac{x}{y}\right)^{d+1}\,.$$

We will show that for all $x \in \mathbb{N}_0$, $y \in \mathbb{N}$, there exists some $\widehat{x} \in \mathbb{N}_0$ such that for all $c \in [0,1]$:

$$g(\widehat{x}, 1, c) \geq g(x,y,c)\,.$$

Fix now $x \in \mathbb{N}_0$ and $y \in \mathbb{N}$. If $y \geq x+1$ then we can choose $\widehat{x} := 0$ to see that for all $c \in [0,1]$ we have $g(0,1,c) = 1 \geq g(x,y,c)$. In the following we therefore assume w.l.o.g. that $y \leq x$.

Define $\widehat{x}$ as the smallest integer such that $g(\widehat{x},1,0) \geq g(x,y,0)$, that is,

$$\widehat{x} := \left\lceil \frac{x+1-y}{y} \right\rceil\,.$$

Since $x \geq y$, we can write $x = b_1 \cdot y + b_2$ for some $b_1 \in \mathbb{N}$ and $b_2 \in [y-1]_0$. This shows that

$$\widehat{x} = b_1 - 1 + \left\lceil \frac{b_2+1}{y} \right\rceil = b_1 = \left\lfloor \frac{x}{y} \right\rfloor\,.$$

Hence,

$$g(\widehat{x},1,1) = \left\lceil \frac{x+1}{y} \right\rceil^d - \widehat{x}^{d+1} \geq \left\lceil \frac{x+1}{y} \right\rceil^d - \left(\frac{x}{y}\right)^{d+1} \geq \left(\frac{x+1}{y}\right)^d - \left(\frac{x}{y}\right)^{d+1} = g(x,y,1)\,.$$

Since $g(\widehat{x},1,\cdot)$ is a linear function, this completes the proof. ◻

**Lemma 6.** *Let $d \in \mathbb{N}$. Then,*

$$I := \inf_{(c_1,c_2)\in(0,1)\times\mathbb{R}} \left\{ \frac{c_2}{1-c_1} \,\middle|\, \forall x,y \in \mathbb{N}_0, f \in \mathcal{P}_d : y \cdot f(x+1) \le c_1 \cdot x \cdot f(x) + c_2 \cdot y \cdot f(y) \right\}$$
$$= \inf_{c\in(0,1)} \left\{ \max_{x\in\mathbb{N}_0} \left\{ \frac{(x+1)^d - c \cdot x^{d+1}}{1-c} \right\} \right\}$$

*Proof.* We first note that for any $c \in (0,1)$ the maximum on the right-hand side exists (Lemma 2).
Assume there is a pair $(c_1, c_2) \in (0,1) \times \mathbb{R}$ with

$$\forall x,y \in \mathbb{N}_0, f \in \mathcal{P}_d : y \cdot f(x+1) \le c_1 \cdot x \cdot f(x) + c_2 \cdot y \cdot f(y)\,. \tag{5.7}$$

Since each $f \in \mathcal{P}_d$ is a linear combination of monomials, (5.7) may be simplified such that $f$ ranges only over all monomials of maximum degree $d$. Moreover, when $y = 0$ the condition is fulfilled trivially. Therefore, (5.7) is equivalent to

$$\forall x \in \mathbb{N}_0, y \in \mathbb{N}, r \in [d]_0 : y \cdot (x+1)^r \le c_1 \cdot x^{r+1} + c_2 \cdot y^{r+1}\,,$$

which is the same as

$$\forall x \in \mathbb{N}_0, y \in \mathbb{N}, r \in [d]_0 : c_2 \ge \left(\frac{x+1}{y}\right)^r - c_1 \cdot \left(\frac{x}{y}\right)^{r+1}\,. \tag{5.8}$$

We can now apply Lemma 5 by which (5.8) is equivalent to

$$\forall x \in \mathbb{N}_0, r \in [d]_0 : c_2 \ge (x+1)^r - c_1 \cdot x^{r+1}\,. \tag{5.9}$$

When $x = 0$, we have for all $r \in [d]_0$ that $(x+1)^r - c_1 \cdot x^{r+1} = 1 > 0$. Therefore, using Lemma 1, we can again simplify (5.9) to

$$\forall x \in \mathbb{N}_0 : c_2 \ge (x+1)^d - c_1 \cdot x^{d+1}\,. \tag{5.10}$$

Finally, we get that

$$I = \inf_{(c_1,c_2)\in(0,1)\times\mathbb{R}} \left\{ \frac{c_2}{1-c_1} \,\middle|\, \forall x \in \mathbb{N}_0 : c_2 \ge (x+1)^d - c_1 \cdot x^{d+1} \right\}$$
$$= \inf_{c\in(0,1)} \left\{ \max_{x\in\mathbb{N}_0} \left\{ \frac{(x+1)^d - c \cdot x^{d+1}}{1-c} \right\} \right\}. \qquad \square$$

Recall from the previous remarks for the proof of Theorem 1 that for each $x \in \mathbb{N}_0$, there is some $c \in (0,1)$ so that $x$ becomes a maximizer (on $\mathbb{N}_0$) of the polynomial $x \mapsto (x+1)^d - c \cdot x^{d+1}$. Consequently, inserting any such pair into the right-hand side of Lemma 6 immediately gives an upper bound on the infimum $I$. In the following, we show how to choose such a pair optimally.

**Lemma 7.** *Let $d \in \mathbb{N}$ and define $k := \lfloor \Phi_d \rfloor$. Then,*

$$I := \inf_{c\in(0,1)} \left\{ \max_{x\in\mathbb{N}_0} \left\{ \frac{(x+1)^d - c \cdot x^{d+1}}{1-c} \right\} \right\} = \frac{(k+1)^{2d+1} - k^{d+1}(k+2)^d}{(k+1)^{d+1} - (k+2)^d + (k+1)^d - k^{d+1}}\,.$$

*Proof.* Define $g : (0,1) \times \mathbb{R} \to \mathbb{R}$,

$$g(c,x) := \frac{(x+1)^d - c \cdot x^{d+1}}{1-c} .$$

The proof proceeds in several steps very similar to Lemma 4. Clearly, $g$ is differentiable on $(0,1)\times\mathbb{R}$.

- Assume there is a pair $(\gamma, \xi) \in (0,1) \times \mathbb{N}_0$ with $g(\gamma,\xi) = g(\gamma, \xi+1)$. Then $g(\gamma,\xi) = \max_{x\in\mathbb{N}_0}\{g(\gamma,x)\}$.

  From the mean value theorem, we know that there is a $\xi^* \in (\xi, \xi+1)$ with $\frac{\partial}{\partial x} g(\gamma,x)|_{x=\xi^*} = 0$. By Lemma 2, we get that $g(\gamma,\cdot)$ has its only positive local maximum at $\xi^*$ and that $g(\gamma,\cdot)$ is strictly increasing in $[0,\xi^*)$ and strictly decreasing in $(\xi^*,\infty)$. Clearly, this implies $g(\gamma,\xi) = g(\gamma,\xi+1) = \max_{x\in\mathbb{N}_0}\{g(\gamma,x)\}$.

- There is a $c_1 \in (0,1)$ with $g(c_1,k) = g(c_1,k+1)$

  Solving the equality $g(c_1,k) = g(c_1,k+1)$ for $c_1$ gives

$$\begin{aligned} (k+1)^d - c_1 \cdot k^{d+1} &= (k+2)^d - c_1 \cdot (k+1)^{d+1} \\ \iff c_1 &= \frac{(k+2)^d - (k+1)^d}{(k+1)^{d+1} - k^{d+1}} . \end{aligned} \tag{5.11}$$

  Since both numerator and denominator of this fraction are positive, and since $(k+2)^d < (k+1)^{d+1}$ and $(k+1)^d > k^{d+1}$, we get that $c_1 \in (0,1)$.

Up to this point, we have shown that

$$I \leq \max_{x\in\mathbb{N}_0}\{g(c_1,x)\} = g(c_1,k) = \frac{(k+1)^{2d+1} - k^{d+1}(k+2)^d}{(k+1)^{d+1} - (k+2)^d + (k+1)^d - k^{d+1}} .$$

In the remainder of the proof we show that also "$\geq$" holds, thereby giving an a posteriori explanation why $c_1$ was chosen "optimally".

- $I \geq g(c_1,k)$

  For all $x \in \mathbb{N}_0$ it holds that

$$\begin{aligned} &\frac{\partial}{\partial c} g(c,x) = \frac{(x+1)^d - x^{d+1}}{(1-c)^2} > 0 \iff x < \Phi_d \iff x \leq k \quad \text{and} \\ &\frac{\partial}{\partial c} g(c,x) < 0 \iff x > k . \end{aligned}$$

  Hence, for $c \in (0,c_1)$ we have $g(c,k+1) > g(c_1,k)$. Similarly, for $c \in (c_1,1)$, we have $g(c,k) > g(c_1,k)$. Altogether, it holds for all $c \in (0,1)\setminus\{c_1\}$ that $\max_{x\in\mathbb{N}_0}\{g(c,x)\} > g(c_1,k)$.

  Figure 3 illustrates the selection of $c_1$: Here, the bold curve shows the function $c \mapsto \max_{x\in\mathbb{N}_0}\{g(c,x)\}$. Note that $\Phi_2 \approx 2.148$ and $(\Phi_2+1)^2 = \Phi_2^3 \approx 9.909$. ∎

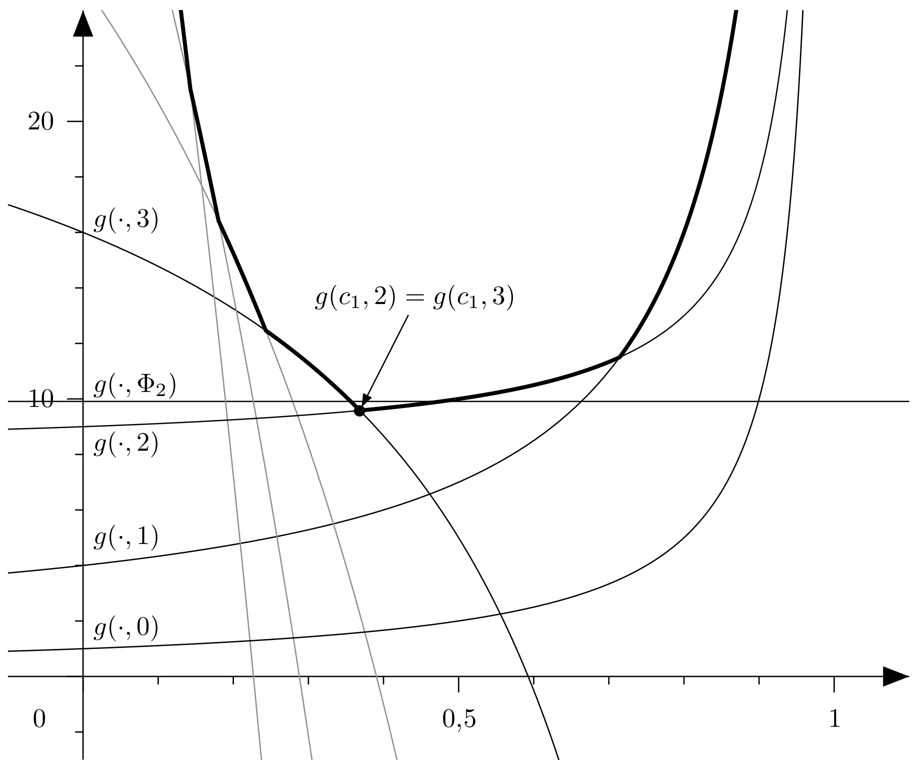


**Figure 3:** Plot of functions $g(\cdot, x)$ for selected values of $x$, when $d = 2$

**Corollary 3.** *The minimum in Theorem 5 is attained for*

$$c_1 = \frac{(k+2)^d - (k+1)^d}{(k+1)^{d+1} - k^{d+1}} \quad \textit{and} \quad c_2 = \frac{(k+1)^{2d+1} - (k+2)^d \cdot k^{d+1}}{(k+1)^{d+1} - k^{d+1}} \,.$$

*Proof.* From the proof of Lemma 6, we know that for any fixed $c_1$ there is an "optimal" $c_2 = \max_{x \in \mathbb{N}_0}\{(x+1)^d - c_1 \cdot x^{d+1}\}$, see (5.10). From the proof of Lemma 7 we get that there is an "optimal" $c_1 \in (0, 1)$, see (5.11). In particular, the infima in the previous lemmas are also minima. Inserting gives the desired result. $\square$

## 6 Conclusion

In this work, we gave exact values for the price of anarchy in (weighted) congestion games with polynomial latency functions of maximum degree $d$. It had been known before that, as $d$ becomes larger, the price of anarchy is growing faster than for the Wardrop model. Nonetheless, it came as a surprise that even in rather simple networks with only quadratic or cubic latency functions, the price of anarchy is already larger as 9 or 40, respectively. Certainly, this result calls for further research: How can networks be *designed* that do not suffer this worst-case loss due to selfish behavior? What is the price of stability, i.e., how do the *best* stable states compare to the system optimum? In particular, the notion of correlated equilibrium (which was considered in this work to formulate our result in the most general way possible) entails the idea of a mediator who can suggest a state that every player would be happy with. If possible, this mediator would, of course, then choose the best possible equilibrium. Also, it would be interesting to transfer or even generalize our technique to other classes of latency functions. Finally, we remark that since the first version of this paper [1] appeared, one interesting follow-up question has been solved: In [13], Gairing and Schoppmann show that the price of anarchy for polynomial latency functions is already attained on restricted parallel links, i.e., for so-called "singleton" congestion games.

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